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THE PRODUCTS OF CONJUGACY CLASSES IN SOME INFINITE SIMPLE GROUPS

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ABSTRACT

Let $H_\nu = S/S^\nu$, where S is the group of all permutations of a set of cardinality $\aleph_\nu$ and S^ν is its subgroup of permutations moving less than $\aleph_\nu$ elements. The infinite simple groups H_ν, $\nu > 0$, have covering number two; that is, $C^2 = H_\nu$ holds for each nonunit conjugacy class $C[M]$. Janko's small group J_1, the only finite simple group with covering number two, satisfies also:

(*) $\qquad C_1 \subseteq C_2 \cdot C_3$ for any nonunit classes C_1, C_2, C_3.

In fact, H_ν $(\nu > 0)$ are the only groups of covering number two where (*) is known to fail. In this paper we determine arbitrary products of classes in H_ν $(\nu > 0)$.

§1. Introduction

The covering number $\mathrm{cn}(G)$ of a group G is defined as the smallest positive integer n for which $C^n = G$ holds for any nonunit conjugacy class (COC) C in G. We write $\mathrm{cn}(G) = \omega$ when no such n exists (so $\mathrm{cn}(G) < \omega$ indicates that such an n exists).

The *general covering number* $\mathrm{gcn}(G)$ of G is defined as the smallest positive integer n such that $C \subseteq C_1 \cdot C_2 \cdots C_n$ holds for any nonunit COCs $C, C_1, \dots, C_n$ in G. We write $\mathrm{gcn}(G) = \omega$ when no such n exists. See [2], [1], [8] for background and more references.

It is not hard to show that $\mathrm{cn}(G) \leqq \mathrm{gcn}(G) + 1$[††] and that $\mathrm{cn}(G) < \omega$ only if G is a simple nonabelian group. If G is a finite nonabelian simple group, then $\mathrm{gcn}(G) < \omega$ (hence $\mathrm{cn}(G) < \omega$); but $\mathrm{cn}(G) = \mathrm{gcn}(G) = \omega$ may hold for an

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[††] In [2], [8] the parameter $\mathrm{ecn}(G)$, defined as the smallest positive integer m for which $G \subseteq C_1 \cdots C_m$ holds for any nonunit COCs $C_1, \dots, C_m$, is treated. It is easily seen that $\mathrm{gcn}(G) < \omega$ iff $\mathrm{ecn}(G) < \omega$ and that then $\mathrm{ecn}(G) = \mathrm{gcn}(G) + 1$.

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infinite simple group ([2], [1], [8]). In cases where the numbers $\mathrm{cn}(G)$, $\mathrm{gcn}(G)$ for a finite nonabelian simple group G are known we have $\mathrm{cn}(G) \leqq \mathrm{gcn}(G)$, and actually $\mathrm{cn}(G) = \mathrm{gcn}(G)$.

Only one finite simple group has covering number two, namely the smallest Janko group J_1. In this case indeed $\mathrm{cn}(J_1) = \mathrm{gcn}(J_1) = 2$ [1]. In [1], [8] it was shown that $\mathrm{cn}(G) = \mathrm{gcn}(G) = 2$ holds frequently for some natural infinite simple groups. Indeed, let S denote the group of all permutations of a set B of cardinality $\aleph_\nu$, $\nu > 0$. Let S^τ denote its normal subgroup of permutations moving less than $\aleph_\tau$ elements, and let $H^\nu_\tau = S^{\tau+1}/S^\tau$, $\tau = 0, \ldots, \nu$. Then H^ν_τ is an infinite simple group and, in fact, $\mathrm{cn}(H^\nu_\tau) = 2$ for $\tau = 0, \ldots, \nu$. Moreover, we have $\mathrm{cn}(H^\nu_\tau) = \mathrm{gcn}(H^\nu_\tau) = 2$ for $0 \leqq \tau < \nu$.

The situation is different for $H^\nu_\nu = S/S^\nu$, which will be briefly denoted as H_ν. It was pointed out in [8] that while $\mathrm{cn}(H_\nu) = 2$, we have $\mathrm{gcn}(H_\nu) > 2$. Presently, the groups H_ν, $\nu > 0$, are the only known groups G with $2 = \mathrm{cn}(G) < \mathrm{gcn}(G) < \omega$. (For $\nu = 0$ we have $\mathrm{cn}(H_0) = \mathrm{gcn}(H_0) = 3$. See [1], [3], [8].)

In this article we study the products of arbitrary COCs of H_ν, $\nu > 0$. It turns out that although $C_1 \subseteq C_2 \cdot C_3$ fails to hold for some nonunit COCs C_1, C_2, C_3 in H_ν, this relation still holds widely. In fact, with one exception, the product of two nonunit COCs of H_ν includes all but at most three nonunit COCs. The exception is provided by the two classes of involutions of H_ν. Those two COCs, with the two COCs of elements of order 3 in H_ν, form a set K of four COCs, with the property that any product of two COCs of K is disjoint from the other two COCs and K is the only set of four COCs with this property. Moreover, whenever C_1, C_2, C_3 are nonunit COCs for which $C_1 \subseteq C_2 \cdot C_3$ fails to hold in H_ν, then at least two of C_1, C_2, C_3 belong to K.

In order to state the results in detail, we anticipate some notations introduced in §2. Let $\langle\hat{2}\rangle$, $\langle\hat{\hat{2}}\rangle$, $\langle\hat{3}\rangle$, $\langle\hat{\hat{3}}\rangle$, $\langle\hat{1}+\hat{2}+\hat{3}\rangle$ denote the respective COCs of $\zeta_1 S^\nu$, $\xi_2 S^\nu$, $\xi_3 S^\nu$, $\xi_4 S^\nu$, $\xi_5 S^\nu$ in H_ν, where $\xi_1, \ldots, \xi_5 \in S$ satisfy the following conditions:

ξ_1 is a fixed-point-free involution; that is, ξ_1 has $\aleph_\nu$ orbits of cardinality 2 and no other orbits.

ξ_2 is an involution with $\aleph_\nu$ fixed points and $\aleph_\nu$ orbits of cardinality two (and no other orbits).

ξ_3 is a fixed-point-free permutation of order 3; that is, ξ_3 has $\aleph_\nu$ orbits of cardinality 3 and no other orbits.

ξ_4 is a permutation of order 3 that has $\aleph_\nu$ fixed points and $\aleph_\nu$ orbits of cardinality 3 (and no other orbits).

ξ_5 is a permutation of order 6, that has $\aleph_\nu$ fixed points, $\aleph_\nu$ orbits of cardinality 2 and $\aleph_\nu$ orbits of cardinality 3 and no other orbits.

Let OD_ν denote the family of all COCs C in H_ν, where C is the COC of ξS^ν for some $\xi \in S$, all of whose orbits are of (finite) odd cardinality.

Write $P(C_1, C_2, C_3)$ for $C_1 \subseteq C_2 \cdot C_3$. Then P is a symmetric 3-place relation on the set of COCs of H_ν (see Proposition 2.3) that completely determines the product of COCs in H_ν, as $C' \cdot C'' = \bigcup_{C \subseteq C' \cdot C''} C$. By $\mathrm{cn}(H_\nu) = 2$, $P(C_1, C_2, C_3)$ holds whenever the nonunit COCs C_1, C_2, C_3 are not distinct. Let us say that a set $\{C_1, C_2, C_3\}$ of COCs is a *P-set* if $P(C_1, C_2, C_3)$ holds, and that it is a *non-P-set* if $P(C_1, C_2, C_3)$ fails. (By the symmetry of P the order in which C_1, C_2, C_3 appear is unimportant.) Thus, the product of any two COCs in H_ν is easily computed, once all non-P-sets are determined. These are determined in

THEOREM 1. *Let $\nu > 0$, and let C_1, C_2, C_3 be nonunit COCs in H_ν. Then $\{C_1, C_2, C_3\}$ is a non-P-set if and only if one of the following three mutually exclusive conditions hold*:

(1) $\{C_1, C_2, C_3\} = \{\langle\hat{2}\rangle, \langle\hat{\hat{2}}\rangle, U\}$, $U \in \mathrm{OD}_\nu$.

(2) $\{C_1, C_2, C_3\} = \{\langle\hat{3}\rangle, \langle\hat{\hat{3}}\rangle, V\}$, $V \in \{\langle\hat{2}\rangle, \langle\hat{\hat{2}}\rangle\}$.

(3) $\{C_1, C_2, C_3\} = \{\langle\hat{2}\rangle, \langle\hat{3}\rangle, \langle\hat{1} + \hat{2} + \hat{3}\rangle\}$.

Let us spell out the actual value of a product of two nonunit COCs in H_ν ($\nu > 0$) as it emerges from Theorem 1. Denote by $\langle\hat{1}\rangle$ the unit COC of H_ν. Then we have

THEOREM A. *Let $\nu > 0$, and let C_1, C_2 be nonunit COCs in H_ν. Then*:

$$C_1 \cdot C_2 = H_\nu \setminus Q(C_1, C_2)$$

where $Q(C_1, C_2) = Q(C_2, C_1)$ is given by:

1. $Q(\langle\hat{2}\rangle, \langle\hat{\hat{2}}\rangle) = \bigcup_{U \in \mathrm{OD}_\nu} U$.
2. $Q(\langle\hat{2}\rangle, \langle\hat{3}\rangle) = \langle\hat{1}\rangle \cup \langle\hat{\hat{2}}\rangle \cup \langle\hat{\hat{3}}\rangle \cup \langle\hat{1} + \hat{2} + \hat{3}\rangle$.
3. $Q(\langle\hat{2}\rangle, \langle\hat{\hat{3}}\rangle) = \langle\hat{1}\rangle \cup \langle\hat{\hat{2}}\rangle \cup \langle\hat{3}\rangle$.
4. $Q(\langle\hat{2}\rangle, \langle\hat{1} + \hat{2} + \hat{3}\rangle) = \langle\hat{1}\rangle \cup \langle\hat{3}\rangle$.
5. $Q(\langle\hat{2}\rangle, U) = \langle\hat{1}\rangle \cup \langle\hat{\hat{2}}\rangle$, $U \in \mathrm{OD}_\nu$.
6. $Q(\langle\hat{\hat{2}}\rangle, \langle\hat{3}\rangle) = \langle\hat{1}\rangle \cup \langle\hat{2}\rangle \cup \langle\hat{\hat{3}}\rangle$.
7. $Q(\langle\hat{\hat{2}}\rangle, \langle\hat{\hat{3}}\rangle) = \langle\hat{1}\rangle \cup \langle\hat{2}\rangle \cup \langle\hat{3}\rangle$.
8. $Q(\langle\hat{\hat{2}}\rangle, U) = \langle\hat{1}\rangle \cup \langle\hat{2}\rangle$, $U \in \mathrm{OD}_\nu$.
9. $Q(\langle\hat{3}\rangle, \langle\hat{\hat{3}}\rangle) = \langle\hat{1}\rangle \cup \langle\hat{2}\rangle \cup \langle\hat{\hat{2}}\rangle$.
10. $Q(\langle\hat{3}\rangle, \langle\hat{1} + \hat{2} + \hat{3}\rangle) = \langle\hat{1}\rangle \cup \langle\hat{2}\rangle$.
11. $Q(C_1, C_2) = \langle\hat{1}\rangle$ *if* $C_1 \neq C_2$ *and* $Q(C_1, C_2)$ *not listed in* 1–10.
12. $Q(C_1, C_2) = \varnothing$ *if* $C_1 = C_2$.

The product of any finite number of COCs in H_ν, $\nu > 0$, is determined by

Theorem A and its following corollary, Theorem B, which follows also from Droste's recent paper ([3], Theorem 2).

THEOREM B. *Let* $\nu > 0$ *and let* C_1, C_2, C_3, C_4 *be any nonunit COCs in* H_ν. *Then*:

(1) $C_1 \cdot C_2 \cdot C_3 \cdot C_4 = H_\nu$,

(2) $C_1 \cdot C_2 \cdot C_3 = H_\nu$ *if* $C_1 \subseteq C_2 \cdot C_3$,

(3) $C_1 \cdot C_2 \cdot C_3 = H_\nu \setminus \langle \hat{1} \rangle$ *if* $C_1 \not\subseteq C_2 \cdot C_3$.

In particular, $\mathrm{gcn}(H_\nu) = 3$.

PROOF. It is enough to prove (2) and (3), as (1) then follows.

(2) Let $C_1 \subseteq C_2 \cdot C_3$. Then $C_1^2 \subseteq C_1 \cdot C_2 \cdot C_3$. But by $\mathrm{cn}(H_\nu) = 2$, $C_1^2 = H_\nu$ and so $C_1 \cdot C_2 \cdot C_3 = H_\nu$.

(3) Let $C_1 \not\subseteq C_2 \cdot C_3$. We show that $C_1 \cdot C_2 \cdot C_3 = H_\nu \setminus \langle \hat{1} \rangle$.

(a) $C_1 \cdot C_2 \cdot C_3 \subseteq H_\nu \setminus \langle \hat{1} \rangle$: Indeed, by $C_1 = C_1^{-1} \not\subseteq C_2 \cdot C_3$ we have $C_1^{-1} \cap C_2 \cdot C_3 = \varnothing$ and so $\langle \hat{1} \rangle \not\subseteq C_1 \cdot C_2 \cdot C_3$.

(b) $H_\nu \setminus \langle \hat{1} \rangle \subseteq C_1 \cdot C_2 \cdot C_3$: Let C be any nonunit COC. We show that $C \subseteq C_1 \cdot C_2 \cdot C_3$. Indeed, let C' be any COC not mentioned in Theorem A 1–10. (for instance, let C be the COC of ξS^ν, where $\xi \in S$ is a permutation all of whose orbits have cardinality 4). Then by Theorem A, $C \subseteq C_1 \cdot C'$ and $C' \subseteq C_2 \cdot C_3$, so $C \subseteq C_1 \cdot C_2 \cdot C_3$. □

Here is an overview of this paper. In §2 we formulate as Theorem 2 a parallel of Theorem 1 that describes the non-P-sets in the family of o-simple COCs — a certain family of COCs in the symmetric group over a countable set. Theorem 2 is stated as a theorem on *types*, these mathematical objects being convenient technical names for classes in the symmetric groups. Theorem 1 is derived from Theorem 2 as follows. We observe that a natural isomorphism exists between the relational structure consisting of the COCs of H_ν with the 3-place relation P, and the structure consisting of the o-simple COCs with the 3-place relation P, when $\aleph_\nu$ is not the sum of countably many smaller cardinals. Then we indicate how Theorem 2 is used to establish Theorem 1 in the remaining cases.

In §3 we show that the sets listed in Theorem 2 are indeed non-P-sets. For sets listed in (1) this follows from the analysis of products of COCs of involutions of a symmetric group S [6]. In fact, we show that if a permutation in S has orbits of odd length only, and is a product of two involutions, then the involutions must be conjugate in S (Proposition 3.2). Similarly, the sets listed in (2) are non-P-sets since the product of two distinct COCs of order 3 in any symmetric group does not contain permutations of order less than 3 (Proposition 3.4).

In §4 we show that the sets not listed in Theorem 2 are indeed P-sets. P-relations established by means of planar Eulerian graphs play a central role here. Proposition 4.2 lists such relations established in [8], and Proposition 4.3 lists some extra relations we need.

In §5 the way to exploit planar Eulerian graphs in this context is explained, and used to establish the relations listed in Proposition 4.3, thereby completing the proof of Theorem 2, and so also of Theorem 1.

§2. From the uncountable to the countable

In this section we show how Theorem 1 follows from Theorem 2 — a parallel Theorem, specifying the non-P-sets of COCs in a certain family of COCs in the symmetric group over a set of cardinality $\aleph_0$.

We use the notation of [8], which we now briefly review. N denotes the set of positive integers and $N^+ = N \cup \{\aleph_0\}$. A *type* t is a cardinal-valued function defined on N^+. For $n \in N^+$, n^* denotes the type defined by $n^*(m) = 1$ if $m = n$, $n^*(m) = 0$ otherwise ($m \in N^+$). Thus $t = \Sigma_{n \in N^+} t(n) \cdot n^*$ holds for any type t, where the sum of an arbitrary set of types and the product of a type by a cardinal number are defined in the natural way.

For every type t we set $|t| = \Sigma_{n \in N^+} t(n) \cdot n$ and call t a *τ-type* if $|t| = \aleph_\tau$. For every ordinal τ we define an equivalence relation $\equiv_\tau$ on types by:

$$t \equiv_\tau s \text{ iff } t = t_0 + r,\ s = s_0 + r \text{ for wome types } t_0,\ s_0,\ r \text{ with } |t_0|, |s_0| < \aleph_\tau.$$

$|B|$ denotes the cardinality of the set B, S_B denotes the group of all permutations of B, and S_B^τ denotes the group of all permutations of B moving less than $\aleph_\tau$ elements. The subscript B is omitted when the context allows. The type $\bar{\xi}$ of $\xi \in S_B$ is defined by requiring that $\bar{\xi}(n)$ is the cardinality of the set of ξ-orbits of cardinality n. Thus, $\bar{\xi}(1)$ is the cardinality of the set of fixed points of ξ and $\bar{\xi}(\aleph_0)$ is that of the set of infinite orbits. If $\bar{\xi} = t$ we say that ξ is a *t-permutation*. Obviously, $|\bar{\xi}| = |B|$ for any $\xi \in S_B$, and whenever t is a type with $|t| = |B|$, there are t-permutations in S_B.

The fundamental role that types play in our context is due to the following well-known basic facts (see [9], 1.3.11; [7], Theorem 4):

PROPOSITION 2.1. *Let $\xi, \eta \in S_B$. Then ξ and η are conjugate in S_B iff $\bar{\xi} = \bar{\eta}$.*

PROPOSITION 2.2. *Let $\xi, \eta \in S_B$. Then ξS_B^τ and ηS_B^τ are conjugate in S_B / S_B^τ iff $\bar{\xi} \equiv_\tau \bar{\eta}$ ($|B| > \aleph_0$).*

We use the letter P to denote several related 3-place relations. The reader will be able to select the right one in a given context. If C_1, C_2, C_3 are COCs in a group G, then $P(C_1, C_2, C_3)$ stands for $C_1 \subseteq C_2 \cdot C_3$. If $g_1, g_2, g_3 \in G$ then $P(g_1, g_2, g_3)$ stands for $[g_1] \subseteq [g_2] \cdot [g_3]$, where $[g] = \{xgx^{-1} : x \in G\}$ is the COC of g in G. When G is a group where $C = C^{-1}$ holds for every COC C, P is a symmetric relation.

For types r, s, t we write $P(r, s, t)$ iff $\xi = \eta\zeta$ holds for some r-permutation ξ, s-permuation η and t-permutation ζ. Thus, $P(r, s, t)$ iff $|r| = |s| = |t|$, and whenever B is a set with $|B| = |r| = |s| = |t|$, then $\xi = \eta\zeta$ holds for some r-permutation ξ, s-permutation η and t-permuation ζ in S_B.

The most useful properties of P as a 3-place relation on types are summarized in

Proposition 2.3 ([8], Lemma 1). *P is symmetric, homogeneous and superadditive; that is:*

(a) *$P(t_1, t_2, t_3)$ iff $P(t_{\theta(1)}, t_{\theta(2)}, t_{\theta(3)})$ for some (equivalently any) permutation θ of* $\{1, 2, 3\}$.

(b) *$P(r, s, t)$ implies $P(kr, ks, kt)$ for any cardinal k.*

(c) *$P(r_i, s_i, t_i)$ for all $i \in I$ imply $P(\Sigma_{i\in I} r_i, \Sigma_{i\in I} s_i, \Sigma_{i\in I} t_i)$.*

By a *simple τ-type* we mean a τ-type t satisfying $t(n) = 0$ or $t(n) = \aleph_\tau$ for all $n \in N^+$ (notice that $t(n) = \aleph_\tau$ must hold for some n by $|t| = \aleph_\tau$). If $|B| = \aleph_\tau$ and $\xi \in S_B$, then ξ is a *simple permutation* and $[\xi]$ is a *simple COC* if $\bar{\xi}$ is a τ-simple type.

We shall identify an ordinal with the set of smaller ordinals and a cardinal number $\aleph_\tau$ with ω_τ, the first ordinal of its cardinality (where ω_0 is abbreviated to ω). Thus $\aleph_\tau = \omega_\tau$ becomes also a canonical set of this cardinality, and we let $H_\nu = S_{\omega_\nu}/S^\nu_{\omega_\nu}$. If t is any τ-type and $\xi \in S_{\omega_\tau}$ is a t-permutation, we let $[t]$ and $\langle t \rangle$ denote the COCs $[\xi]$ in S_{ω_τ} and $[\xi S^\nu_{\omega_\nu}]$ in H_ν respectively.

We say that an ordinal τ has *cofinality* ω and write $\mathrm{cof}(\tau) = \omega$ if τ has a cofinal infinite countable subset. We shall need this notion only to abbreviate the statement that $\aleph_\tau$ is a sum of countably many smaller cardinals — which may happen only if $\tau = 0$ or $\mathrm{cof}(\tau) = \omega$ ([5], IV. 3.9, p. 134). We can now state

Proposition 2.4. *Let $\nu > 0$, $\mathrm{cof}(\nu) \neq \omega$. Then for every COC C in H_ν there is a unique simple ν-type t such that $C = \langle t \rangle$.*

Proof. We first note that at most one such t exists, for, by Proposition 2.2, if t and t' are two distinct simple ν-types, then $\langle t \rangle \neq \langle t' \rangle$.

Now let $C = [\xi S^\nu_{\omega_\nu}]$ be any COC in H_ν, and define a type t by $t(n) = 0$ if $\bar{\xi}(n) < \aleph_\nu$, $t(n) = \aleph_\nu$ otherwise. Since $\nu > 0$ and $\mathrm{cof}(\nu) \neq \omega$, $\aleph_\nu$ is not the sum of countably many smaller ordinals. Thus, we must have $t(n)=\aleph_\nu$ for some $n \in N^+$ and so t is a simple ν-type; moreover $t \equiv_\nu \bar{\xi}$, whence $C=\langle t\rangle$. □

For a COC C in H_ν, $0<\nu$, $\mathrm{cof}(\nu)\neq\omega$, let t_C denote the unique simple ν-type t for which $C=\langle t_C\rangle$.

PROPOSITION 2.5. *Let $\nu>0$, $\mathrm{cof}(\nu)\neq\omega$ and let C_1, C_2, C_3 be COCs in H_ν. Then $P(C_1,C_2,C_3)$ iff $P(t_{C_1},t_{C_2},t_{C_3})$.*

PROOF. Assume first $P(t_{C_1},t_{C_2},t_{C_3})$. This means that $[t_{C_1}]\subseteq[t_{C_2}]\cdot[t_{C_3}]$, which implies that $\langle t_{C_1}\rangle\subseteq\langle t_{C_2}\rangle\cdot\langle t_{C_3}\rangle$, i.e. $C_1\subseteq C_2\cdot C_3$, and so $P(C_1,C_2,C_3)$.

Assume now that $P(C_1,C_2,C_3)$ and let ξ, η, $\zeta\in S_{\omega_\nu}=S$ satisfy

$$\xi S^\nu=\eta S^\nu\cdot\zeta S^\nu,\quad [\xi S^\nu]=C_1,\quad [\eta S^\nu]=C_2,\quad [\zeta S^\nu]=C_3.$$

We may further assume that $\bar{\xi}=t_{C_1}$, $\bar{\eta}=t_{C_2}$, $\bar{\zeta}=t_{C_3}$. By our assumption, $\xi=\eta\zeta\sigma$ holds in S, where $\sigma\in S^\nu$. Let A_0 denote the set of points moved by σ. Then $|A_0|<\aleph_\nu$. Define A_{n+1} from A_n by setting

$$A_{n+1}=A_n\cup\eta(A_n)\cup\eta^{-1}(A_n)\cup\zeta(A_n)\cup\zeta^{-1}(A_n).$$

Then $A=\bigcup_{n\in\omega}A_n$ satisfies $|A|<\aleph_\nu$, and $\xi\,|\,A$, $\eta\,|\,A$, $\zeta\,|\,A\in S_A$. Let

$$\xi'=\xi\,|\,(\omega_\nu\setminus A),\quad \eta'=\eta\,|\,(\omega_\nu\setminus A),\quad \zeta'=\zeta\,|\,(\omega_\nu\setminus A).$$

Then ξ', η', $\zeta'\in S_{\omega_\nu\setminus A}$, $\xi'=\eta'\zeta'$ and $\bar{\xi}'=t_{C_1}$, $\bar{\eta}'=t_{C_2}$, $\bar{\zeta}'=t_{C_3}$. Thus, $P(t_{C_1},t_{C_2},t_{C_3})$. □

For any nonzero type t define a simple 0-type t^0 by setting $t^0(n)=0$ if $t(n)=0$ and $t^0(n)=\aleph_0$ otherwise. Obviously, $t\to t^0$ is a one-to-one mapping if t varies on simple ν-types. Let $\Sigma_\nu=\langle\Sigma_\nu,P\rangle$ be the relational structure with domain the set Σ_ν of all simple ν-types, and the 3-place relation P (restricted to Σ_ν). Our next proposition shows that $t\to t^0$ is actually an isomorphism of Σ_ν with Σ_0.

PROPOSITION 2.6. *Let r, s, t be simple ν-types. Then $P(r,s,t)$ iff $P(r^0,s^0,t^0)$.*

PROOF. Assume first $P(r^0,s^0,t^0)$. Then by homogeneity of P (Proposition 2.3) also $P(\aleph_\nu\cdot r^0,\ \aleph_\nu\cdot s^0,\ \aleph_\nu\cdot t^0)$. But $\aleph_\nu\cdot r^0=r$, $\aleph_\nu\cdot s^0=s$, $\aleph_\nu\cdot t^0=t$ and so $P(r,s,t)$.

Assume now $P(r,s,t)$, and let ξ, η, ζ be permutations of a set B of cardinality $\aleph_\nu$ satisfying $\xi=\eta\zeta$ and $\bar{\xi}=r$, $\bar{\eta}=s$, $\bar{\zeta}=t$. For each l with $r(l)>0$, m with $s(m)>0$, n with $t(n)>0$ let $C_l\subseteq B$, $D_m\subseteq B$, $E_n\subseteq B$ be a countable union of ξ-orbits of cardinality l, of η-orbits of cardinality m, and of ζ-orbits of cardinality n, respectively. Let

$$A_0 = \Big(\bigcup_{r(l)>0} C_l\Big) \cup \Big(\bigcup_{s(m)>0} D_m\Big) \cup \Big(\bigcup_{t(n)>0} E_n\Big).$$

Then $|A_0| = \aleph_0$. Let

$$A_{n+1} = A_n \cup \eta(A_n) \cup \eta^{-1}(A_n) \cup \zeta(A_n) \cup \zeta^{-1}(A_n),$$

and let $A = \bigcup_{n\in\omega} A_n$. Then $|A| = \aleph_0$ and $\xi' = \xi \mid A$, $\eta' = \eta \mid A$, $\zeta' = \zeta \mid A$ are permutations of A satisfying $\xi' = \eta'\zeta'$, $\bar{\xi}' = r^0$, $\bar{\eta}' = s^0$ and $\bar{\zeta}' = t^0$. Thus $P(r^0, s^0, t^0)$. □

Let $\mathbf{COC}(H_\nu) = \langle \mathrm{COC}(H_\nu), P\rangle$ denote the relational structure consisting of the set $\mathrm{COC}(H_\nu)$ consisting of all COCs of H_ν with the three place relation P (restricted to $\mathrm{COC}(H_\nu)$). Combining Propositions 2.4, 2.5, 2.6 we have

Proposition 2.7. *Let $\nu > 0$, $\mathrm{cof}(\nu) \neq \omega$. Then $\mathbf{COC}(H_\nu) = \langle \mathrm{COC}(H_\nu), P\rangle$ and $\Sigma_0 = \langle \Sigma_0, P\rangle$ are isomorphic; in fact, the mapping $C \to t_C^0$ is an isomorphism of $\mathbf{COC}(H_\nu)$ with Σ_0.*

The structure Σ_0 — and thereby the structures Σ_ν for any ordinal ν — is completely determined once the relation P is specified. By symmetry of P (Proposition 2.1) it is enough to specify all sets $\{r, s, t\}$ of simple 0-types for which P fails. This is the content of Theorem 2. For $n \in N$ let $\tilde{n}$, $\tilde{\tilde{n}}$ be the simple 0-types defined by:

$\tilde{n} = \aleph_0 \cdot n^*$,

$\tilde{\tilde{n}} = \tilde{1} + \tilde{n} = \aleph_0 \cdot (1^* + n^*)$.

Let od_0 denote the set of 0-types satisfying $t(2n) = 0$ for all $n \in N^+$.

Theorem 2. *Let r, s, t be simple 0-types. Then $P(r, s, t)$ fails if and only if r, s, t satisfy one of the following three mutually exclusive conditions:*

(1) $\{r, s, t\} = \{\tilde{2}, \tilde{\tilde{2}}, u\}$, $u \in \mathrm{od}_0$,

(2) $\{r, s, t\} = \{\tilde{3}, \tilde{\tilde{3}}, v\}$, $v \in \{\tilde{2}, \tilde{\tilde{2}}\}$,

(3) $\{r, s, t\} = \{\tilde{2}, \tilde{3}, \tilde{1} + \tilde{2} + \tilde{3}\}$.

The proof of Theorem 2 is given in §3, 4, 5.

Let now $\nu > 0$, and set for $n \in N^+$

$\hat{n} = \aleph_\nu \cdot n^*$,

$\hat{\hat{n}} = \hat{1} + \hat{n} = \aleph_\nu (1^* + n^*)$.

Let od_ν denote the set of ν-types t satisfying $t(2n) = 0$ for all $n \in N^+$, and let $\mathrm{OD}_\nu = \{\langle t\rangle : t \in \mathrm{od}_\nu\}$.

With this notation, Theorem 1 for $\nu > 0$, $\mathrm{cof}(\nu) \neq \omega$ is an immediate consequence of Theorem 2 and Proposition 2.7. We now briefly indicate a way to prove Theorem 1 in the case $\mathrm{cof}(\nu) = \omega$, leaving the details to the reader.

Assume that $\mathrm{cof}(\nu)=\omega$. Then every type t with $t(n)<\aleph_\nu$ for all $n\in N^+$, $\Sigma_{n\in N^+}t(n)=\aleph_\nu$ defines a *nonsimple* class $\langle t\rangle$ in H_ν; indeed, $\langle t\rangle\neq\langle s\rangle$ for all simple ν-types s, by Proposition 2.2. Thus Proposition 2.4 fails, due to the presence of nonsimple classes, and a substitute for it is needed.

Let us say that a ν-type t is a *special ν-type* if it satisfies the following conditions:

1. $t(n)<\aleph_\nu$ for all $n\in N^+$, $\Sigma_{n>1}t(n)=\aleph_\nu$.
2. $t(1)=t(2)=t(3)=0$.
3. If $t(n)>0$ then $t(n)\geqq\aleph_0$.
4. $\Sigma_{n\in N^+}t(2n)\in\{0,\aleph_\nu\}$, $\Sigma_{n\in N}t(2n+1)\in\{0,\aleph_\nu\}$.

Proposition 2.8. *Let* $\mathrm{cof}(\nu)=\omega$. *Then every nonunit COC C in H_ν satisfies* $C=\langle t\rangle$, *where t satisfies one of the following three mutually exclusive conditions*:

(1) *t is a simple ν-type.*

(2) *t is a special ν-type.*

(3) $t=t_1+t_2$, *where t_1 is a simple ν-type and t_2 is a special ν-type.*

Next one shows that all triples of COCs of H_ν listed in Theorem 1 are indeed non-P-sets. Negating this assumption, one arrives at a counterexample to Theorem 2, using Proposition 2.8 and an argument of the kind provided for Proposition 2.5.

Finally, Theorem 2 is used to show that if C_1, C_2, C_3 are COCs in H_ν, and $\{C_1,C_2,C_3\}$ is not a triple listed in Theorem 1, then $P(C_1,C_2,C_3)$. This follows from:

Proposition 2.9. *Let* $\mathrm{cof}(\nu)=\omega$. *Let r, s, t be nonunit ν-types satisfying the conditions of Proposition* 2.8. *Assume further*:

[1) $\{r,s,t\}\neq\{\hat{2},\hat{\hat{2}},u\}$, $u\in\mathrm{od}_\nu$,

(2) $\{r,s,t\}\neq\{\hat{3},\hat{\hat{3}},u\}$, $v\in\{\hat{2},\hat{\hat{2}}\}$,

(3) $\{r,s,t\}\neq\{\hat{2},\hat{3},\hat{1}+\hat{2}+\hat{3}\}$.

Then $P(r,s,t)$.

Outline of Proof. 1. It is enough to show that one can write $r=\Sigma_{i\in I}r_i$, $s=\Sigma_{i\in I}s_i$, $t=\Sigma_{i\in I}t_i$, where r_i, s_i, t_i are simple 0-types and $\{r_i,s_i,t_i\}$ is not listed in Theorem 2 for each $i\in I$.

Indeed, then $P(r,s,t)$ follows from Theorem 2 and Proposition 2.3(c).

2. We may assume that each of r, s, t is either a simple ν-type or a special ν-type.

Indeed, we have $t=t+t$ whenever t is either a simple ν-type or a special ν-type. Thus we can write $(r,s,t)=(r',s',t')+(r'',s'',t'')$, where each of r', r'', s', s'', t', t'' is either a simple ν-type or a special ν-type.

3. We may assume at least one of r, s, t — say t — to be a special ν-type.

Indeed, if r, s, t are simple ν-types then $P(r,s,t)$ follows by 1.

4. We may assume $\{r,s\} \neq \{\hat{2},\hat{\hat{2}}\}$.

Indeed, assume $\{r,s\} = \{\hat{2},\hat{\hat{2}}\}$. Since t is a special ν-type, $\Sigma_{n\in N^+} t(2n) \in \{0, \aleph_\nu\}$. If this sum is 0, then $t \in \mathrm{od}_\nu$ and so (1) is violated. Thus $\Sigma_{n\in N^+} t(2n) = \aleph_\nu$. $P(r,s,t)$ follows by 1.

5. We may assume $r \neq s$.

Indeed, otherwise $P(r,s,t)$ follows by 1.

6. If the assumptions listed in 2–5 hold, then $P(r,s,t)$.

Indeed, since t is a special ν-type, $t(n) = 0$ for $n = 1,2,3$, and since $r \neq s$, $\{r,s\} \neq \{\hat{2},\hat{\hat{2}}\}$ we may assume, say, $\Sigma_{n>2} s(n) = \aleph_\nu$. $P(r,s,t)$ follows by 1. □

§3. Theorem 2 lists only non-P-sets

In this section we show that $P(r,s,t)$ fails for 0-types r, s, t if $\{r,s,t\}$ satisfies one of the three conditions listed in Theorem 2. We say that $\{r,s,t\}$ is a *P-set* (a *non-P-set*) to denote that $P(r,s,t)$ holds (fails), i.e. that an r-permutation can (cannot) be represented as a product of an s-permutation by a t-permutation.

PROPOSITION 2.1. *$P(\tilde{2},\tilde{\tilde{2}},u)$ fails for $u \in \mathrm{od}_0$.*

Proposition 3.1 follows from the following proposition, which states that if the product of two COCs of involutions in some symmetric group contains a permutation all of whose orbits are of odd cardinality, then the two COCs must be equal.

PROPOSITION 3.2. *Let w be a type satisfying $w(2n) = 0$ for all $n \in N^+$. If $P(w, m_1 \cdot 1^* + m_2 \cdot 2^*, n_1 \cdot 1^* + n_2 \cdot 2^*)$ then $m_1 = n_1$ and $m_2 = n_2$.*

PROOF. This proposition follows from [6]. (If $|w| < \aleph_0$ use [6], Theorem 2.6. If $|w| \geqq \aleph_0$ use [6], Theorem 2.1 (or 2.1') and Theorem A.1.) We give an independent proof:

Let ξ, η, $\zeta \in S$ satisfy $\xi = \eta\zeta$, $\eta^2 = \zeta^2 = 1$ and assume that all ξ-orbits have odd cardinality. For each $a \in B$ let $2n_a - 1$ be the cardinality of the ξ-orbit containing a. Then $\xi^{n_a-1}(a) = \xi^{-n_a}(a)$ holds for each $a \in B$ and $\xi^{-1} = \zeta\eta$. Thus

$$\xi^{n_a-1}(a) = (\xi^{-1})^{n_a}(a) = (\zeta\eta)^{n_a}(a) = \zeta(\eta\zeta)^{n_a-1}\eta(a) = \zeta\xi^{n_a-1}\eta(a).$$

Hence $\eta(a) = a$ if and only if $\zeta(\xi^{n_a-1}(a)) = \xi^{n_a-1}(a)$. Thus $a \to \xi^{n_a-1}(a)$ is a permutation of B mapping the set of fixed points of η onto the set of fixed points of ζ, and so $\bar{\eta}(1) = \bar{\zeta}(1)$, and $\bar{\eta}(2) = \bar{\zeta}(2)$. □

PROPOSITION 3.3. *$P(\tilde{3},\tilde{\tilde{3}},v)$ fails for $v \in \{\tilde{2},\tilde{\tilde{2}}\}$.*

Proposition 3.3 is a consequence of the following proposition, which states that the product of two distinct COCs of elements of order 3 in any symmetric group does not contain a permutation of order less than 3.

PROPOSITION 3.4. *Assume $P(l_1 \cdot 1^* + l_2 \cdot 2^*,\ m_1 \cdot 1^* + m_3 \cdot 3^*,\ n_1 \cdot 1^* + n_3 \cdot 3^*)$ where $l_1, l_2, m_1, m_3, n_1, n_3$ are any cardinal numbers. Then $m_1 = n_1$ and $m_3 = n_3$.*

Proposition 3.4 is proved using

LEMMA 3.5. *Let G be a group with unit 1, and let $g, h \in G$ satisfy $g^3 = h^3 = (gh)^2 = 1$. Then $gh^{-1}ghg^{-1}h = 1$.*

PROOF. Note that $(hg)^2 = 1$, as $hg = g^{-1}(gh)g$. Thus, $gh = h^{-1}g^{-1}$ and $gh^{-1}ghg^{-1}h = gh^{-1}h^{-1}g^{-1}g^{-1}h = ghgh = gg^{-1}h^{-1}h = 1$, where the second equality follows from $g^3 = h^3 = 1$ and the third from $(hg)^2 = 1$. □

PROOF OF PROPOSITION 3.4. Let ξ, η, ζ be permutations of a set B, satisfying $\xi = \eta\zeta$, $\xi^2 = \eta^3 = \zeta^3 = 1$. Let $\mathrm{Fix}(\eta)$, $\mathrm{Fix}(\zeta) \subseteq B$ denote the sets of fixed points of η, ζ, respectively. We shall show that the permutation $\eta\zeta^{-1} \in S_B$ maps $\mathrm{Fix}(\eta)$ onto $\mathrm{Fix}(\zeta)$, and so $\bar{\eta}(1) = \bar{\zeta}(1)$, $\bar{\eta}(3) = \bar{\zeta}(3)$ and Proposition 3.4 is established.

It follows from Lemma 3.5 that:

$$\eta\zeta^{-1}\eta\zeta\eta^{-1} = \zeta^{-1}, \qquad \zeta\eta^{-1}\zeta\eta\zeta^{-1} = \eta^{-1}.$$

Let $a \in \mathrm{Fix}(\eta)$; that is, $\eta(a) = a$. Then $\zeta\eta^{-1}\zeta\eta\zeta^{-1}(a) = \eta^{-1}(a) = a$, so $\zeta(\eta\zeta^{-1}(a)) = \eta\zeta^{-1}(a)$, i.e., $\eta\zeta^{-1}(a) \in \mathrm{Fix}(\zeta)$. Similarly, if $\zeta(b) = b$ then $\eta(\zeta\eta^{-1}(b)) = \zeta\eta^{-1}(b)$, i.e., $\zeta\eta^{-1}(b) \in \mathrm{Fix}(\eta)$. Thus

$$\eta\zeta^{-1}(\mathrm{Fix}(\eta)) \subseteq \mathrm{Fix}(\zeta) \quad \text{and} \quad \zeta\eta^{-1}(\mathrm{Fix}(\zeta)) \subseteq \mathrm{Fix}(\eta).$$

By $\zeta\eta^{-1} = (\eta\zeta^{-1})^{-1}$, we have $\eta\zeta^{-1}(\mathrm{Fix}(\eta)) = \mathrm{Fix}(\zeta)$. □

PROPOSITION 3.6. *$P(\tilde{2},\tilde{3},\tilde{1}+\tilde{2}+\tilde{3})$ fails.*

This proposition follows from

PROPOSITION 3.7. *Assume $P(l_2 \cdot 2^*,\ m_1 \cdot 1^* + m_3 \cdot 3^*,\ n_1 \cdot 1^* + n_2 \cdot 2^* + n_3 \cdot 3^*)$, where $l_2, m_1, m_3, n_1, n_2, n_3$ are cardinal numbers. Then $n_1 \leqq m_1$.*

PROOF OF PROPOSITION 3.7. Let ξ, η, ζ be permutations of a set B satisfying

$$\xi = \eta\zeta, \quad \bar{\xi} = l_2 \cdot 2^*, \quad \bar{\eta} = m_1 \cdot 1^* + m_3 \cdot 3^*, \quad \bar{\zeta} = n_1 \cdot 1^* + n_2 \cdot 2^* + n_3 \cdot 3^*.$$

We shall show that $\zeta\eta^{-1} \in S_B$ maps fixed points of ζ to fixed points of η and conclude that $n_1 \leqq m_1$, thus establishing Proposition 3.6.

Let $\zeta(a) = a$, and let $d = \zeta\eta^{-1}(a)$. We shall show that $\eta(d) = d$. Since this is clear when $d = a$, we assume further that $d \neq a$. Let $b = \eta(a)$, $c = \eta(b)$. $\eta(d) = d$ will be the last of a list of claims that we now establish.

1. $a \neq b \neq c \neq a$, $\eta(c) = a$ *and* $\zeta(c) = d$

Indeed, $b = \eta(a) \neq a$, as otherwise $\eta^{-1}(a) = a$ and so $d = \zeta\eta^{-1}(a) = a$, contradicting our assumption. The rest follows by $\eta^3 = 1$ and $\zeta\eta^{-1}(a) = d$.

2. $\xi(b) = a$

Indeed, $\xi(a) = \eta\zeta(a) = \eta(a) = b$, so by $\xi^2 = 1$, $\xi(b) = a$.

3. $\zeta(b) = c$

Indeed, $\zeta(b) = \eta^{-1}\xi(b) = \eta^{-1}(a) = c$ by 1.

4. $d \neq b \neq c \neq d$, *and* $\zeta(b) = c$, $\zeta(c) = d$, $\zeta(d) = b$

Indeed, if $d = b$ then $\xi(c) = \eta(\zeta(c)) = \eta(d) = \eta(b) = c$. But then $\bar{\xi}(1) > 0$, contradicting $\bar{\xi} = l_2 \cdot 2^*$. Thus $d \neq b$.

If $d = c$ then $\xi(c) = \eta(\zeta(c)) = \eta(d) = \eta(c) = a$, which is impossible by 1 and 2($b \neq c$ by 1). The rest follows from 1, 3 and $\zeta^3(b) = b$ (as $\bar{\zeta}(n) = 0$ for $n > 3$).

5. $\xi(c) = d$

Indeed, $\xi(d) = \eta(\zeta(d)) = \eta(b) = c$, so by $\xi^2 = 1$, $\xi(c) = d$.

6. $\eta(d) = d$

Indeed, $\eta(d) = \eta(\zeta(c)) = \xi(c) = d$, by 5. □

Propositions 3.1, 3.3 and 3.6 show that whenever r, s, t are 0-simple types satisfying one of the conditions (1), (2), (3) of Theorem 2, then $\{r, s, t\}$ is a non-P-set, i.e., Theorem 2 lists only non-P-sets.

§4. Theorem 2 lists all non-P-sets

In this section we show that $P(r, s, t)$ holds for all simple 0-types r, s, t, if $\{r, s, t\}$ is not listed in Theorem 2. Let us put:

$$L_1 = \{\{\tilde{2}, \tilde{\tilde{2}}, u\} : u \in \mathrm{od}_0\},$$

$$L_2 = \{\{\tilde{3}, \tilde{\tilde{3}}, v\} : v \in \{\tilde{2}, \tilde{\tilde{2}}\}\},$$

$$L_3 = \{\{\tilde{2}, \tilde{3}, \tilde{1} + \tilde{2} + \tilde{3}\}\},$$

$$L = L_1 \cup L_2 \cup L_3.$$

We shall prove:

THEOREM 4.1. *Let r, s, t be simple 0-types, $\{r,s,t\}\not\in L$. Then $P(r,s,t)$.*

Before proving Theorem 4.1 we state in two propositions the P-relations we shall need. We shall use freely the properties of P listed in Proposition 2.3, namely symmetry, homogeneity and superadditivity.

PROPOSITION 4.2. *The following relations hold:*

(1) $P(r,s,s)$, $\quad s\neq\tilde{1}$, r, s *simple* 0-*types*,
(2) $P(\tilde{l},\tilde{m},\tilde{n})$, $\quad l,m,n>1$, $l,m,n\in N^+$,
(3) $P(\tilde{\tilde{l}},\tilde{\tilde{m}},\tilde{\tilde{n}})$, $\quad l,m,n>1$, $l,m,n\in N^+$,
(4) $P(\tilde{l},\tilde{m},\tilde{\tilde{n}})$, $\quad l,m,n>2$, $l,m,n\in N^+$,
(5) $P(\tilde{l},\tilde{\tilde{m}},\tilde{\tilde{n}})$, $\quad l,m,n>2$, $l,m,n\in N^+$,
(6) $P(\tilde{2},\tilde{m},\tilde{\tilde{n}})$, $\quad m,n>3$, $m,n\in N^+$,
(7) $P(\tilde{2},\tilde{\tilde{m}},\tilde{\tilde{n}})$, $\quad m,n>3$, $m,n\in N^+$,
(8) $P(\tilde{\tilde{2}},\tilde{m},\tilde{n})$, $\quad m,n>3$, $m,n\in N^+$,
(9) $P(\tilde{\tilde{2}},\tilde{\tilde{m}},\tilde{n})$, $\quad m,n>3$, $m,n\in N^+$,
(10) $P(\tilde{2},\tilde{\tilde{2}},\widetilde{2n})$, $\quad n\in N^+$.

PROOF. For (1), (2), (4), (6), see [8] Lemmas 3, 4, Proposition 5.0 and for (10) see [9] 10.1.17 or [8] Proposition 3.5. (3) follows from (2) by $(\tilde{\tilde{l}},\tilde{\tilde{m}},\tilde{\tilde{n}})=(\tilde{l},\tilde{m},\tilde{n})+(\tilde{1},\tilde{1},\tilde{1})$. (5), (7), (9) follow from (4), (6), (8) by $(r,\tilde{\tilde{m}},\tilde{\tilde{n}})=(r,\tilde{m},\tilde{\tilde{n}})+(r,\tilde{\tilde{m}},\tilde{n})$ whenever $r+r=r$. (8) is proved very much like (4) ([8], Proposition 5.1, case 1). $\square$

Suitable planar Eulerian graphs will help us establish in §5

PROPOSITION 4.3. *Let $n\in N^+$, $3<n$, then:*

(11) $P(\tilde{2},\tilde{3},\tilde{\tilde{n}})$,
(12) $P(\tilde{2},\tilde{\tilde{3}},\tilde{n})$,
(13) $P(\tilde{\tilde{2}},\tilde{3},\tilde{n})$,
(14) $P(\tilde{2},\tilde{\tilde{3}},\tilde{\tilde{n}})$,
(15) $P(\tilde{\tilde{2}},\tilde{3},\tilde{\tilde{n}})$,
(16) $P(\tilde{\tilde{2}},\tilde{\tilde{3}},\tilde{n})$.

We shall prove (11)–(13) in section 5. (14)–(16) follows from (11)–(13) (see proof of Proposition 4.2).

PROOF OF THEOREM 4.1. Assume that r, s, t are nonunit simple 0-types, and that $\{r,s,t\}\not\in L$. We shall prove that $P(r,s,t)$. We first show that some extra assumptions may be made, eliminating the cases violating these assumptions.

(*i*) *We may assume* $r \neq s \neq t \neq r$. Indeed, if $r = s$ or $s = t$ or $t = r$ then $P(r, s, t)$ by (1).

(ii) We may assume $\{\tilde{2}, \tilde{\tilde{2}}\} \not\subseteq \{r, s, t\}$. Indeed, otherwise $\{r, s, t\} = \{\tilde{2}, \tilde{\tilde{2}}, w\}$, where by $\{r, s, t\} \notin L$, $w(2n) = \aleph_0$ for some n. But by (10) $P(\tilde{2}, \tilde{\tilde{2}}, \widetilde{2n})$ and so $P(\tilde{2}, \tilde{\tilde{2}}, w)$ by (1) and $(\tilde{2}, \tilde{\tilde{2}}, w) = (\tilde{2}, \tilde{\tilde{2}}, \widetilde{2n}) + (\tilde{2}, \tilde{2}, w)$; thus $P(r, s, t)$.

Let us write $r = r' + r''$, $s = s' + s''$, $t = t' + t''$ where:

$$r' = \sum_{n<4} r(n) \cdot n^*, \qquad r'' = \sum_{4 \leqq n} r(n) \cdot n^*,$$

$$s' = \sum_{n<4} s(n) \cdot n^*, \qquad s'' = \sum_{4 \leqq n} s(n) \cdot n^*,$$

$$t' = \sum_{n<4} t(n) \cdot n^*, \qquad t'' = \sum_{4 \leqq n} t(n) \cdot n^*.$$

Let $M = \{\tilde{2}, \tilde{\tilde{2}}, \tilde{3}, \tilde{\tilde{3}}, \tilde{2} + \tilde{3}, \tilde{1} + \tilde{2} + \tilde{3}\}$ be the set of nonunit simple 0-types r satisfying $r = r'$.

(*iii*) *We may assume* $r'' \neq 0$ *or* $s'' \neq 0$ *or* $t'' \neq 0$.

PROOF. We shall show that $P(r, s, t)$ holds otherwise. Indeed, assume $r = r'$, $s = s'$, $t = t'$. Thus, $r, s, t \in M$. By (ii) we may further assume $\{\tilde{2}, \tilde{\tilde{2}}\} \not\subseteq \{r, s, t\}$. Thus, $P(r, s, t)$ follows from (1) and the following relations:

$$\begin{array}{lll}
P(\tilde{2}, \tilde{3}, \tilde{2} + \tilde{3}) & \text{by} & (\tilde{2}, \tilde{3}, \tilde{2} + \tilde{3}) = (\tilde{2}, \tilde{3}, \tilde{2}) + (\tilde{2}, \tilde{3}, \tilde{3}), \\
P(\tilde{2}, \tilde{\tilde{3}}, \tilde{2} + \tilde{3}) & \text{by} & (\tilde{2}, \tilde{\tilde{3}}, \tilde{2} + \tilde{3}) = (\tilde{2}, \tilde{\tilde{3}}, \tilde{2}) + (\tilde{2}, \tilde{3}, \tilde{3}), \\
P(\tilde{2}, \tilde{\tilde{3}}, \tilde{1} + \tilde{2} + \tilde{3}) & \text{by} & (\tilde{2}, \tilde{\tilde{3}}, \tilde{1} + \tilde{2} + \tilde{3}) = (\tilde{2}, \tilde{\tilde{3}}, \tilde{\tilde{3}}) + (\tilde{2}, \tilde{3}, \tilde{2}), \\
P(\tilde{\tilde{2}}, \tilde{3}, \tilde{2} + \tilde{3}) & \text{by} & (\tilde{\tilde{2}}, \tilde{3}, \tilde{2} + \tilde{3}) = (\tilde{\tilde{2}}, \tilde{3}, \tilde{3}) + (\tilde{2}, \tilde{3}, \tilde{2}), \\
P(\tilde{\tilde{2}}, \tilde{3}, \tilde{1} + \tilde{2} + \tilde{3}) & \text{by} & (\tilde{\tilde{2}}, \tilde{3}, \tilde{1} + \tilde{2} + \tilde{3}) = (\tilde{\tilde{2}}, \tilde{3}, \tilde{\tilde{2}}) + (\tilde{2}, \tilde{3}, \tilde{3}), \\
P(\tilde{\tilde{2}}, \tilde{\tilde{3}}, \tilde{2} + \tilde{3}) & \text{by} & (\tilde{\tilde{2}}, \tilde{\tilde{3}}, \tilde{2} + \tilde{3}) = (\tilde{\tilde{2}}, \tilde{3}, \tilde{3}) + (\tilde{2}, \tilde{\tilde{3}}, \tilde{2}), \\
P(\tilde{\tilde{2}}, \tilde{\tilde{3}}, \tilde{1} + \tilde{2} + \tilde{3}) & \text{by} & (\tilde{\tilde{2}}, \tilde{\tilde{3}}, \tilde{1} + \tilde{2} + \tilde{3}) = (\tilde{\tilde{2}}, \tilde{\tilde{3}}, \tilde{\tilde{3}}) + (\tilde{2}, \tilde{3}, \tilde{2}), \\
P(\tilde{3}, \tilde{\tilde{3}}, \tilde{2} + \tilde{3}) & \text{by} & (\tilde{3}, \tilde{\tilde{3}}, \tilde{2} + \tilde{3}) = (\tilde{3}, \tilde{\tilde{3}}, \tilde{3}) + (\tilde{3}, \tilde{3}, \tilde{2}), \\
P(\tilde{3}, \tilde{\tilde{3}}, \tilde{1} + \tilde{2} + \tilde{3}) & \text{by} & (\tilde{3}, \tilde{\tilde{3}}, \tilde{1} + \tilde{2} + \tilde{3}) = (\tilde{2}, \tilde{\tilde{3}}, \tilde{\tilde{3}}) + (\tilde{3}, \tilde{3}, \tilde{2}), \\
P(\tilde{\tilde{3}}, \tilde{2} + \tilde{3}, \tilde{1} + \tilde{2} + \tilde{3}) & \text{by} & (\tilde{\tilde{3}}, \tilde{2} + \tilde{3}, \tilde{1} + \tilde{2} + \tilde{3}) = (\tilde{\tilde{3}}, \tilde{2} + \tilde{3}, \tilde{\tilde{3}}) + (\tilde{3}, \tilde{2}, \tilde{2}).
\end{array}$$

This completes the proof of (ii).

We actually proved the following restriction of Theorem 4.1:

PROPOSITION 4.4. *Let* $r = r'$, $s = s'$, $t = t'$. *If* $\{r, s, t\} \notin L$ *then* $P(r, s, t)$.

(iv) We may assume at least two of r'', s'', t'' *to be nonzero.*

PROOF. We show that if $r'' = s'' = 0$, $t'' \neq 0$ then $P(r, s, t)$. By assumption $r = r'$, $s = s'$ and so $r, s \in M$. By (ii) we may further assume $\{r, s\} \neq \{\tilde{2}, \tilde{\tilde{2}}\}$. Thus, $P(r, s, t'')$ follows from (2), (12), (13), (16) and $(r, s, t'') = \Sigma_{t''(n) \neq 0}(r, s, \tilde{n})$. It follows that $P(r, s, t)$ if $t' = 0$, and so we assume $t' \neq 0$. If $t' = \tilde{1}$ then $P(r, s, t)$ follows from (3), (11), (14), (15) and $(r, s, t) = \Sigma_{t''(n) \neq 0}(r, s, \tilde{\tilde{n}})$. Thus we assume $t' \in M$. If $P(r, s, t')$ then $P(r, s, t)$ follows from $(r, s, t) = (r, s, t') + (r, s, t'')$, so we may assume that $P(r, s, t')$ fails. Hence, by Proposition 4.4, $\{r, s, t'\} \in L$. Thus, $P(r, s, t)$ follows from $(r, s, t) = (r, s, t' + \tilde{n}) + (r, s, t'')$, where $t''(n) > 0$, and

PROPOSITION 4.5. *Let* $\{r, s, t\} \in L$. *Then* $P(r, s, t + \tilde{n})$ *holds whenever* $\{r, s, t + \tilde{n}\} \notin L$.

PROOF. By Proposition 4.4 and (ii) we may assume $n > 3$ and $\{r, s\} \neq \{\tilde{2}, \tilde{\tilde{2}}\}$. The proposition follows from (1), (2), (4), (11)–(16) and the following identities:

$$(\tilde{2} + \tilde{n}, \tilde{\tilde{2}}, w) = (\tilde{n}, \tilde{\tilde{2}}, w) + (\tilde{2}, \tilde{2}, w) \quad (w = \tilde{3}, \tilde{\tilde{3}}),$$

$$(\tilde{2}, \tilde{\tilde{2}} + \tilde{n}, w) = (\tilde{2}, \tilde{\tilde{n}}, 2) + (\tilde{2}, \tilde{2}, w) \quad (w = \tilde{3}, \tilde{\tilde{3}}),$$

$$(\tilde{3} + \tilde{n}, \tilde{\tilde{3}}, w) = (\tilde{n}, \tilde{\tilde{3}}, w) + (\tilde{3}, \tilde{3}, w) \quad (w = \tilde{2}, \tilde{\tilde{2}}),$$

$$(\tilde{3}, \tilde{\tilde{3}} + \tilde{n}, w) = (\tilde{3}, \tilde{\tilde{n}}, w) + (\tilde{3}, \tilde{3}, w) \quad (w = \tilde{2}, \tilde{\tilde{2}}),$$

$$(\tilde{3}, \tilde{\tilde{3}}, w + \tilde{n}) = (\tilde{3}, \tilde{\tilde{3}}, \tilde{n}) + (\tilde{3}, \tilde{3}, w) \quad (w = \tilde{2}, \tilde{\tilde{2}}),$$

$$(\tilde{2} + \tilde{n}, \tilde{3}, \tilde{1} + \tilde{2} + \tilde{3}) = (\tilde{n}, \tilde{3}, \tilde{\tilde{3}}) + (\tilde{2}, \tilde{3}, \tilde{2}),$$

$$(\tilde{2}, \tilde{3} + \tilde{n}, \tilde{1} + \tilde{2} + \tilde{3}) = (\tilde{2}, \tilde{n}, \tilde{\tilde{3}}) + (\tilde{2}, \tilde{3}, \tilde{2}),$$

$$(\tilde{2}, \tilde{3}, \tilde{1} + \tilde{2} + \tilde{3} + \tilde{n}) = (\tilde{2}, \tilde{3}, \tilde{\tilde{n}}) + (\tilde{2}, \tilde{3}, \tilde{2}) + (\tilde{2}, \tilde{3}, \tilde{3}). \qquad \square$$

The proof of (iv) is complete.

We now state the following corollary of Propositions 4.2 and 4.3, whose straightforward verification is left to the reader:

PROPOSITION 4.6. *Let* r, s, t *be nonunit simple* 0*-types,* $s(2) = s(3) = t(2) = 0$. *Then* $P(r, s, t)$.

COROLLARY. *Let* r, s, t *be nonunit simple* 0*-types, with* $r'' \neq 0$, $s'' \neq 0$, $t'' \neq 0$. *Then* $P(r, s, t)$.

PROOF. Indeed, $(r, s, t) = (r, s'', t'') + (r'', s, t'') + (r'', s'', t)$. Hence by Proposition 4.6, $P(r, s, t)$. □

In view of this corollary, the proof of Theorem 4.1 will be complete if we can show that

(*v*) *We may assume* $r'' \neq 0$, $s'' \neq 0$, $t'' \neq 0$.

PROOF. In view of (iv) it is enough to show $P(r, s, t)$, given $r'' = 0$, $s'' \neq 0$, $t'' \neq 0$. Thus, $r \in M$. By Proposition 4.6 we have $P(r, s'', t'')$. We show first that we may assume $s', t' \in M$.

Assume first that $t' \notin M$; that is, $t' = 0$ or $t' = \tilde{1}$. Then by Proposition 4.6 we have $P(r, s'', t)$. If also $s' \notin M$, we have by Proposition 4.6 $P(r, s, t)$, so we may assume $s' \in M$. Thus $r, s' \in M$. We now distinguish three cases.

1. $r = \tilde{2}$, $s' = \tilde{\tilde{2}}$. Then $P(r, s, t)$ by $(r, s, t) = (\tilde{2}, \tilde{2} + s'', t) = (\tilde{2}, \tilde{1} + s'', t) + (\tilde{2}, \tilde{2}, t)$, as by Proposition 4.6 we have $P(\tilde{2}, \tilde{1} + s'', t)$ and by (1) $P(\tilde{2}, \tilde{2}, t)$.

2. $r = \tilde{\tilde{2}}$, $s' = \tilde{2}$. Then $P(r, s, t)$ by $(r, s, t) = (\tilde{\tilde{2}}, \tilde{2} + s'', t) = (\tilde{\tilde{2}}, s'', t) + (\tilde{2}, \tilde{2}, t)$, again by Proposition 4.6 and (1).

3. $\{r, s'\} \neq \{\tilde{2}, \tilde{\tilde{2}}\}$. Then by Proposition 4.5 $P(r, s', \tilde{n})$ holds for any $n > 3$, so if $t''(n) > 0$, $P(r, s, t)$ follows from $(r, s, t) = (r, s', \tilde{n}) + (r, s''; t)$.

Assume finally that $s', t' \in M$. Then $P(r, s, t)$ holds if $P(r, s', t')$ does, by $P(r, s'', t'')$ and $(r, s, t) = (r, s', t') + (r, s'', t'')$. So assume that $P(r, s', t')$ fails. Then by Proposition 4.4 $\{r, s', t'\} \in L$. Distinguish three cases:

1. $\{r, s', t'\} = \{\tilde{2}, \tilde{\tilde{2}}, u\}$, $u \in \{\tilde{3}, \tilde{\tilde{3}}\}$. Then $P(r, s, t)$ is established as follows:

If $(r, s', t') = (\tilde{2}, \tilde{\tilde{2}}, u)$, by $(r, s, t) = (\tilde{2}, \tilde{1} + s'', t'') + (\tilde{2}, \tilde{2}, t)$.

If $(r, s', t') = (\tilde{\tilde{2}}, \tilde{2}, u)$, by $(r, s, t) = (\tilde{\tilde{2}}, s'', t'') + (\tilde{2}, \tilde{2}, t)$.

If $(r, s', t') = (u, \tilde{2}, \tilde{\tilde{2}})$, by $(r, s, t) = (r, s'', \tilde{1} + t'') + (r, \tilde{2}, \tilde{2})$.

2. $\{r, s, t\} = \{\tilde{3}, \tilde{\tilde{3}}, v\}$, $v \in \{\tilde{2}, \tilde{\tilde{2}}\}$. Then $P(r, s, t)$ is established as follows:

If $(r, s', t') = (\tilde{3}, \tilde{\tilde{3}}, v)$, by $(r, s, t) = (\tilde{3}, \tilde{1} + s'', t'') + (\tilde{3}, \tilde{3}, t)$.

If $(r, s', t') = (\tilde{\tilde{3}}, \tilde{3}, v)$, by $(r, s, t) = (\tilde{\tilde{3}}, s'', t'') + (\tilde{3}, \tilde{3}, t)$.

If $(r, s', t') = (v, \tilde{\tilde{3}}, \tilde{3})$, by $(r, s, t) = (v, \tilde{1} + s'', t'') + (v, \tilde{3}, \tilde{3})$.

3. $\{r, s', t'\} = \{\tilde{2}, \tilde{3}, \tilde{1} + \tilde{2} + \tilde{3}\}$. Then $P(r, s, t)$ is established as follows:

If $(r, s', t') = (\tilde{2}, \tilde{3}, \tilde{1} + \tilde{2} + \tilde{3})$, by $(r, s, t) = (\tilde{2}, s'', \tilde{1} + t'') + (\tilde{2}, \tilde{3}, \tilde{2} + \tilde{3})$.

If $(r, s', t') = (\tilde{3}, \tilde{1} + \tilde{2} + \tilde{3}, \tilde{2})$, by $(r, s, t) = (\tilde{3}, \tilde{1} + s'', t'') + (\tilde{3}, \tilde{2} + \tilde{3}, \tilde{2})$.

If $(r,s,t)=(\tilde{1}+\tilde{2}+\tilde{3},\tilde{2},\tilde{3})$, by $(r,s,t)=(\tilde{1}+\tilde{2}+\tilde{3},s'',t'')+(\tilde{2}+\tilde{3},\tilde{2},\tilde{3})$.

This completes the proof of (v), and thereby the proof of Theorem 4.1.

§5. Some more planar Eulerian graphs

In this section we establish Proposition 4.3, thereby completing the proof of Theorem 4.1, and of Theorem 2. We use planar Eulerian graphs to establish the P-relations (11), (12), (13) listed in the proposition. A brief description of this method, introduced in [8], follows.

By a bicolored planar Eulerian graph (BPEG) we mean a planar Eulerian graph (PEG) G with a proper black and white coloring of the G-regions — the plane regions it defines. (That is, two G-regions whose closures share an edge of G are colored differently.) With each BPEG G one associates three types, b_G, d_G, w_G, where $b_G(l)$ is the number of black G-regions bounded by l G-edges, $d_G(m)$ is the number of G-vertices of degree $2m$ and $w_G(n)$ is the number of white G-regions bounded by n G-edges. In [8] it is shown that

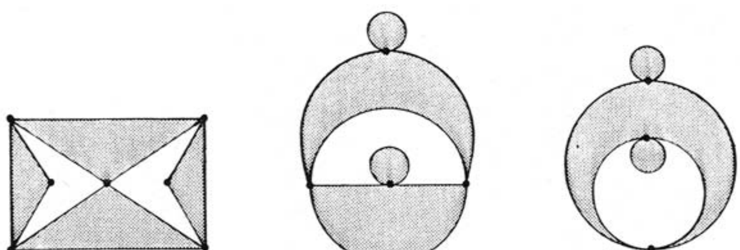
THEOREM 5.1. *Let G be any BPEG. Then $P(b_G, d_G, w_G)$.*

With this theorem at hand, we simultaneously prove graphically (11)–(13), restated as:

THEOREM 5.2. *Let $n \in N^+$, $4 \leqq n$. Then:*
$(11)_n$ $P(\tilde{2},\tilde{3},\tilde{\tilde{n}})$,
$(12)_n$ $P(\tilde{2},\tilde{\tilde{3}},\tilde{n})$,
$(13)_n$ $P(\tilde{\tilde{2}},\tilde{2},\tilde{n})$.

PROOF. We treat separately the cases $n=4$, $n=5$, $6 \leqq n < \aleph_0$ and $n=\aleph_0$.

Case 1. $n=4$, see Fig. 1.
By (a) we have $P(4\cdot 3^*, 2\cdot 1^* + 5\cdot 2^*, 3\cdot 4^*)$, and by



(a) (b) (c)

Fig. 1.

$$(\tilde{3},\tilde{\tilde{2}},\tilde{4}) = \aleph_0\cdot(4\cdot 3^*, 2\cdot 1^* + 5\cdot 2^*, 3\cdot 4^*)$$

we have $P(\tilde{3},\tilde{\tilde{2}},\tilde{4})$, and so by symmetery of P, $P(\tilde{\tilde{2}},\tilde{3},\tilde{4})$, i.e., $(13)_4$ holds.

By (b) we have $P(2\cdot(1^*+3^*), 4\cdot 2^*, 2\cdot 4^*)$ and so $P(\tilde{2},\tilde{\tilde{3}},\tilde{4})$ holds, i.e. $(12)_4$ holds.

By (c) $P(2\cdot 1^* + 4^*, 3\cdot 2^*, 2\cdot 3^*)$ holds, and so $P(\tilde{2},\tilde{3},\tilde{\tilde{4}})$ holds, i.e. $(11)_4$ holds.

Case 2. $n = 5$, see Fig. 2.

By (a), $P(3\cdot 5^*, 3\cdot 1^* + 6\cdot 2^*, 5\cdot 3^*)$, and so $P(\tilde{\tilde{2}},\tilde{3},\tilde{5})$ by

$$(\tilde{3},\tilde{\tilde{2}},\tilde{5}) = \aleph_0\cdot(3\cdot 5^*, 3\cdot 1^* + 6\cdot 2^*, 5\cdot 3^*)$$

and the symmetry of P. Thus $(13)_5$ holds

By (b), $P(2\cdot 1^* + 6\cdot 3^*, 10\cdot 2^*, 4\cdot 5^*)$ and so $P(\tilde{2},\tilde{\tilde{3}},\tilde{5})$, i.e. $(12)_5$ holds.

By (c), $P(4\cdot 3^*, 6\cdot 2^*, 2\cdot(1^*+5^*))$ and so $P(\tilde{2},\tilde{3},\tilde{\tilde{5}})$, i.e. $(11)_5$ holds.

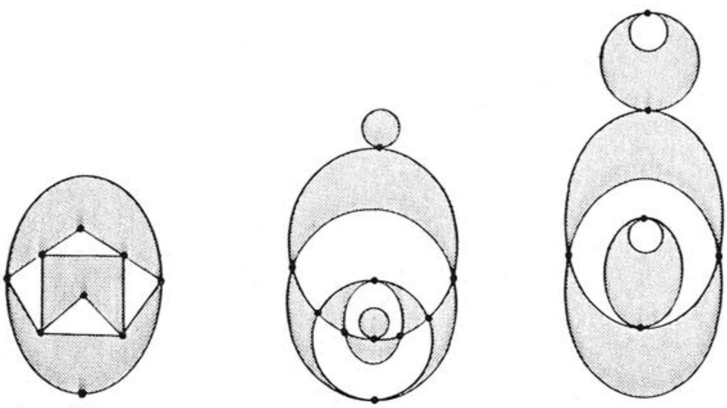


Fig. 2.

Case 3. $6 \leqq n < \aleph_0$.

We establish graphically the case $n = 6$, see Fig. 3.

By (a), $P(\tilde{3}, 1^* + \tilde{2}, \tilde{6})$ and so by $(\tilde{3},\tilde{\tilde{2}},\tilde{6}) = \aleph_0\cdot(\tilde{3}, 1^* + \tilde{2}, \tilde{6})$ and symmetry of P, $P(\tilde{\tilde{2}},\tilde{3},\tilde{6})$, i.e. $(13)_6$ holds.

By (b), $P(1^* + \tilde{3}, \tilde{2}, \tilde{6})$ and so $P(\tilde{2},\tilde{\tilde{3}},\tilde{6})$, i.e. $(12)_6$ holds.

By (c), $P(\tilde{3}, \tilde{2}, 1^* + \tilde{6})$ and so $P(\tilde{2},\tilde{3},\tilde{\tilde{6}})$, i.e. $(11)_6$ holds.

For $6 < n < \aleph_0$ similar BPEGs are easily drawn (and an inductive construction for all n can be described; see [8], proof of Proposition 5.1).

Case 4. $n = \aleph_0$, see Fig. 4.

By (a), $P(\tilde{3},\tilde{\tilde{2}},\aleph_0^*)$ and so $P(\tilde{\tilde{2}},\tilde{3},\tilde{\aleph}_0)$ by $(\tilde{3},\tilde{\tilde{2}},\tilde{\aleph}_0) = \aleph_0\cdot(\tilde{3},\tilde{\tilde{2}},\aleph_0^*)$ and symmetry of P. Thus $(13)_{\aleph_0}$ holds.

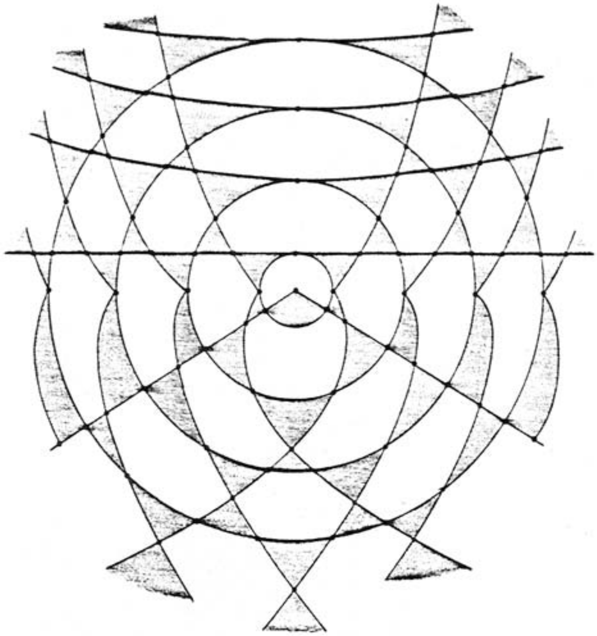


Fig. 3(a).

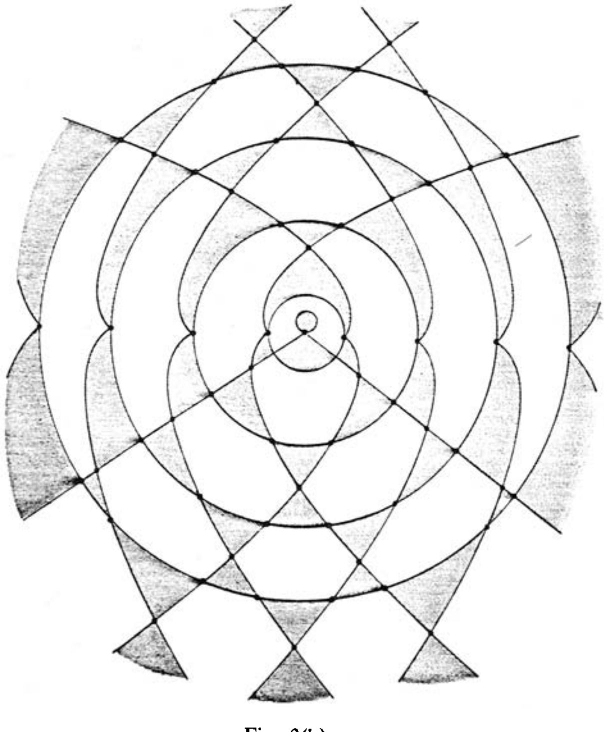


Fig. 3(b).

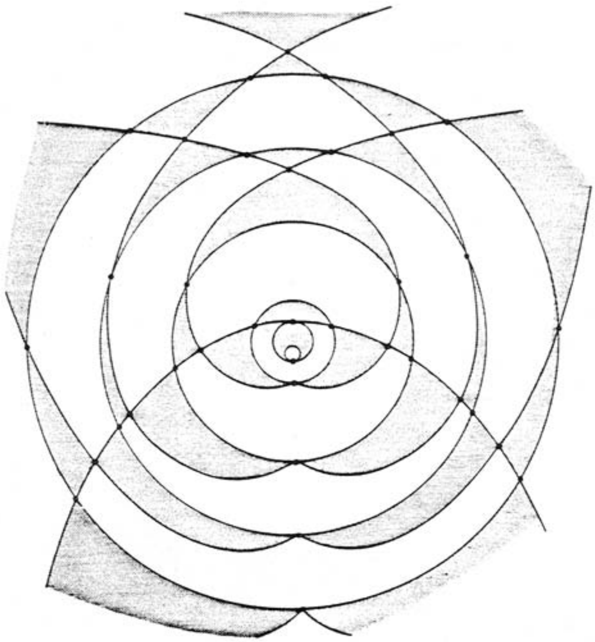
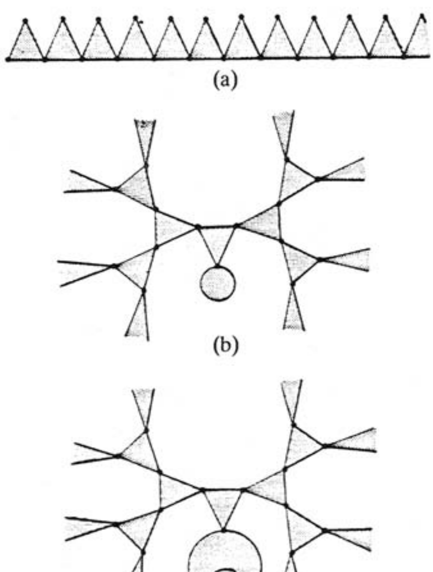


Fig. 3(c).



(a)

(b)

(c)

Fig. 4.

By (b), $P(1^* + \tilde{3}, \tilde{2}, \tilde{\aleph}_0)$ and so $P(\tilde{2}, \tilde{\tilde{3}}, \tilde{\aleph}_0)$ and $(12)_{\aleph_0}$ holds.
By (c), $P(\bar{3}, \tilde{2}, 1^* + \aleph_0^*)$ and so $P(\tilde{2}, \tilde{3}, \tilde{\tilde{\aleph}}_0)$ and $(11)_{\aleph_0}$ holds.

This completes the proof of Theorem 5.2, and the proof of Theorem 2 with it.

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